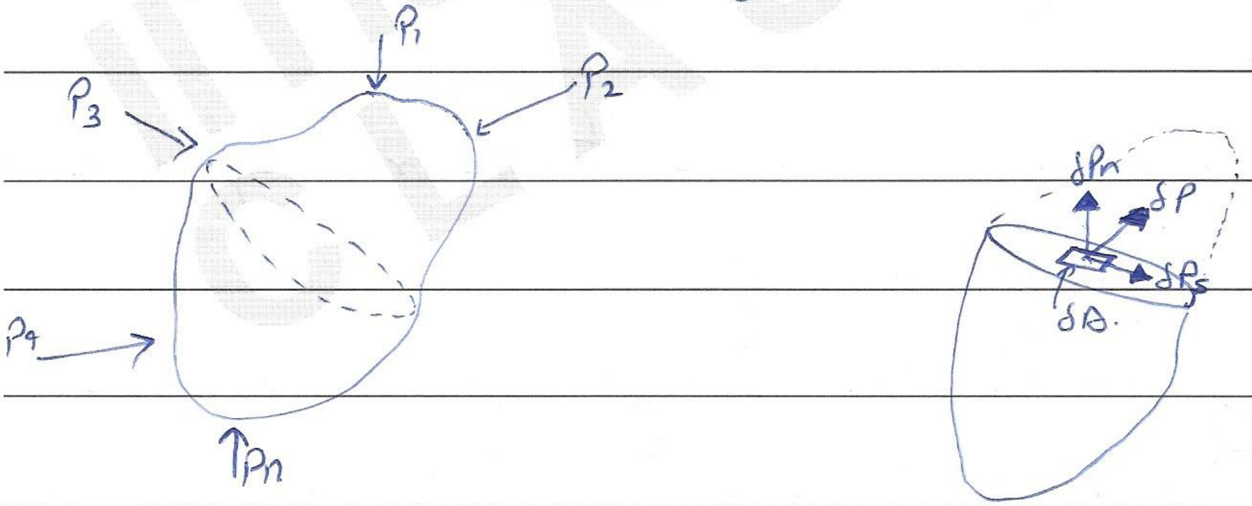
Basic Elasticity

Stress:

When a body undergoes deformation under the application of external force, a restoring/resistance force is induce within the body. The intensity of this restoring force, i.e., restoring force per unit area is termed as stress.

$$\text{Stress} = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}$$



Under complex loading, for any given small area, δA, the resultant force can be at any inclination. Thus the restoring force (hence stress) is resolved in two components:

1> Normal stress $\sigma = \lim_{\delta A \to 0} \frac{\delta P_N}{\delta A}$ = normal to plane

2> Shear stress $\tau = \lim_{\delta A \to 0} \frac{\delta P_S}{\delta A}$ = Parallel to plane.

- Stress is a Tensor quantity (2^{nd} order tensor), i.e., it depends on magnitude, direction and plane in which it acts.
- Normal stress (σ) can be tensile or compressive in nature depending on loading.
- Generally, normal stress pointing away from plane is considered as tensile stress (+ve) while normal stress pointing towards the plane is considered as compressive stress (−ve).

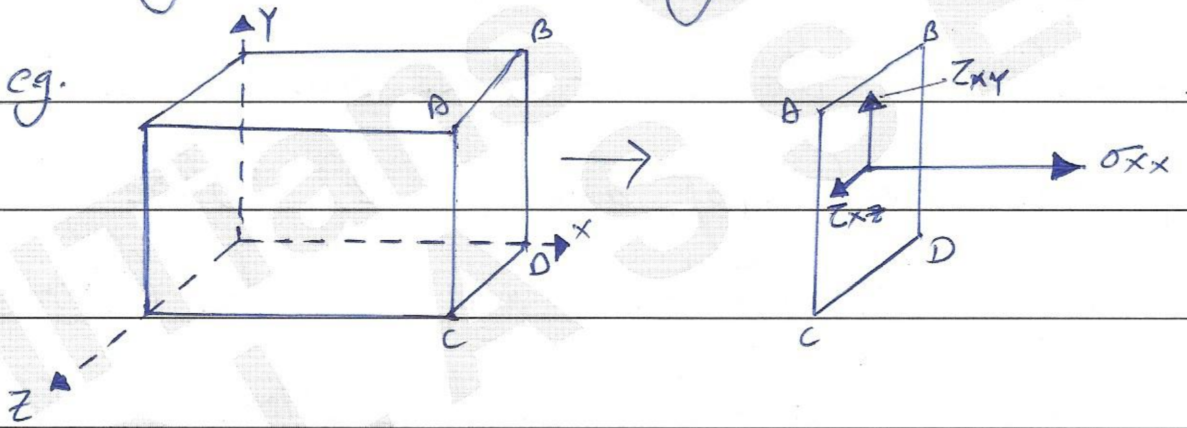
- Normal stress is also termed as direct stress.

Notation and Direction of Stresses:

In tensorial notation the stress is generally termed as σ_{ij} or τ_{ij} → two suffix for 2nd order Tensor

where i indicates plane in which it acts &
j indicates direction of action.

For eg.



In above figure plane ABCD is 'x-plane'. There is one normal force and two shear stress in a plane of '3-D structure'. The normal stress in plane ABCD is notated as σ_{xx}. Similarly, shear stress τ_{xy} in y-direction and τ_{xz} in z-direction.

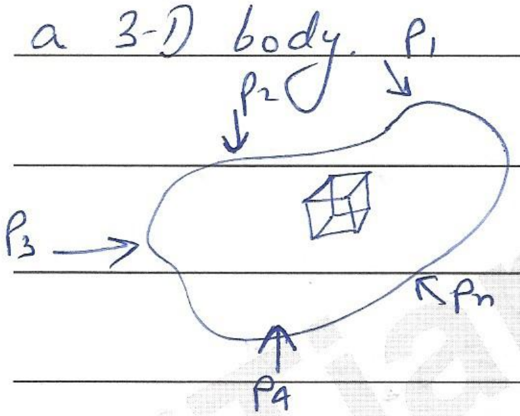
- Some time, σ_{nn} is also termed with σ_n (single σ_n).

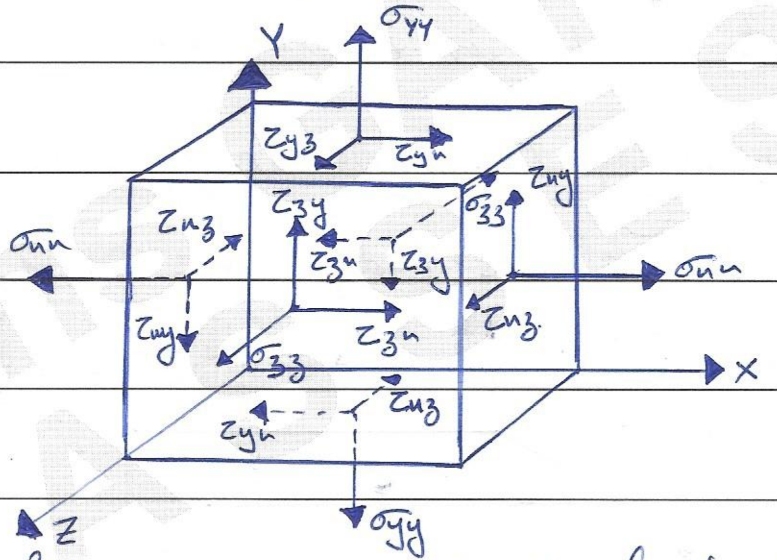
Direction: - The normal stresses are defined as positive when they are directed away from their related surface.

- The direction of shear stress depend on the coordinate system considered.
  - If the tensile stress is in positive direction of the axis (say x), the shear stress are positive in other two positive direction of axis (Y & Z). While if tensile stress in opposite direction of axis (-X), the positive shear stress are in direction opposite to positive direction of axis (-Y & -Z).

Stresses in 3-D Body

If any 3D body is subjected to external load, it undergo deformation which induce strains and stresses. Consider a small (infinitesimal) particle of a 3-D body.





There are 3 planes in a cubical partical (X, Y, Z).

In each plane → one normal stress and two shear stress.

∴ There are total 9 stresses in a 3D body.

In terms of matrix,

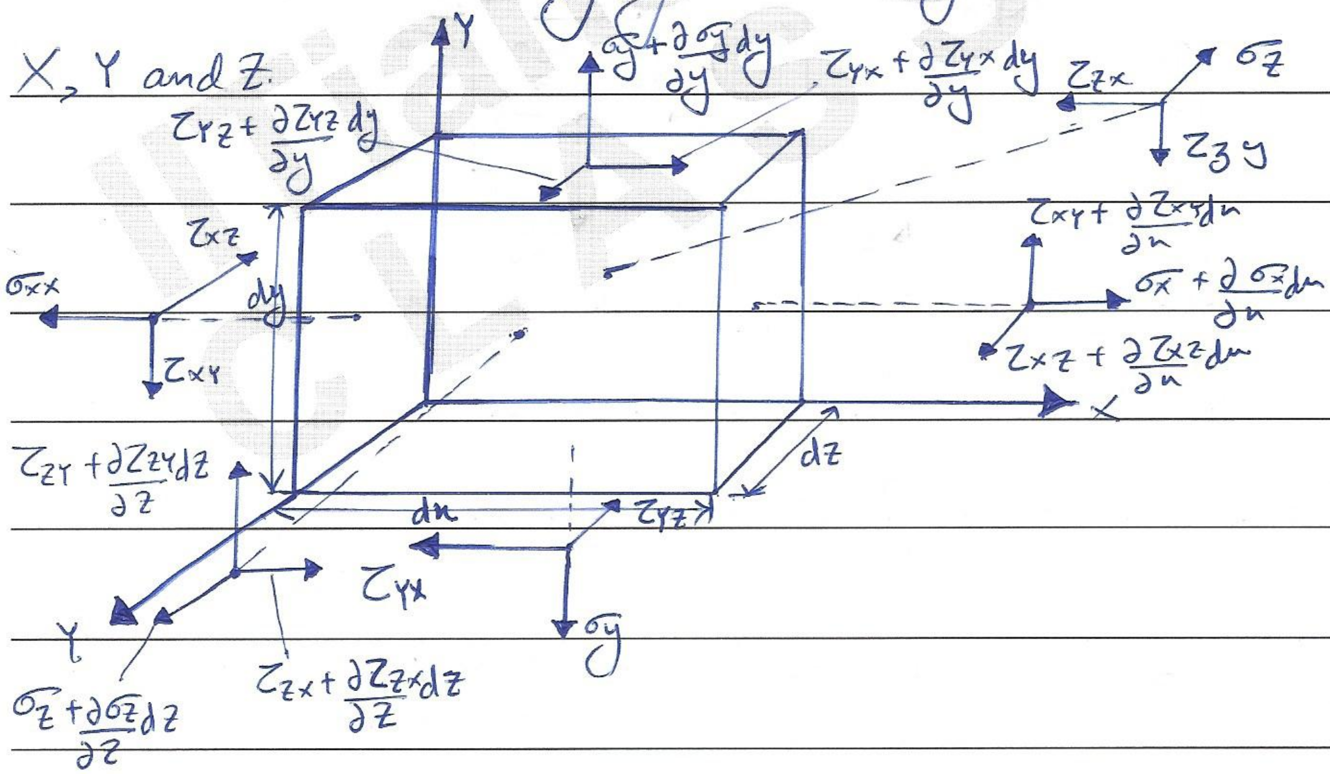
$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}_{3\times 3}$$

* Row indicates plane
* Column indicates direction

Equation of Equilibrium:

Considering again the infinitesimal element of a 3D body subjected to external loading. Assume the body to be in equilibrium under the external loading, hence the small element is also in equilibrium.

Considering the variation of stresses along the element as well. Let the body force in x, y and z direction is X, Y and Z.



Taking moment about an axis through centre of element and parallel to z-axis

$$(\tau_{xy}\,dy\,dz)\frac{dx}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right)dy\,dz\cdot\frac{dx}{2} - \tau_{yx}\,dx\,dz\,\frac{dy}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dx\,dz\cdot\frac{dy}{2} = 0$$

Ignoring higher order term and dividing by $dx\,dy\,dz$:

$\Rightarrow \boxed{\tau_{xy} = \tau_{yx}}$

Similarly $\Rightarrow \boxed{\tau_{xz} = \tau_{zx}}$ and $\boxed{\tau_{yz} = \tau_{zy}}$

→ Complementary nature of shear stress.

- A shear stress acting on a given plane ($\tau_{xy}, \tau_{xz}, \tau_{yz}$) is always accompanied by an equal complementary shear stress ($\tau_{yx}, \tau_{zx}, \tau_{zy}$) acting on a plane perpendicular to the given plane and in opposite sense.

Now considering force equilibrium in X-direction

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy\, dz - \sigma_x\, dy\, dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\, dz - \tau_{yx}\, dx\, dz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\, dy - \tau_{zx}\, dx\, dy + X\, dx\, dy\, dz = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0}$$

Using complementary shear stress, i.e., $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0$$

The above eqn. of equilibrium must be satisfied at all interior points in a deformable body under a 3-D force system.

Strain

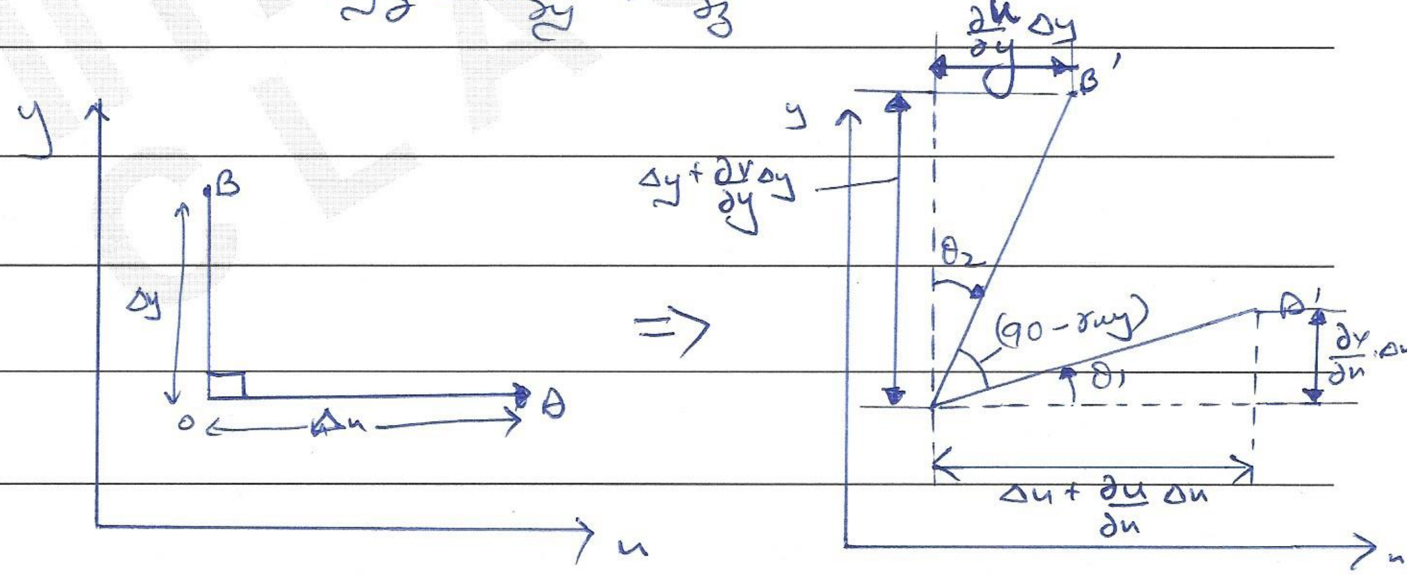
Strain is defined as change in length to original length.

- Normal strain $\xrightarrow{\text{due to}}$ Normal stress.
- Shear strain $\xrightarrow{\text{due to}}$ shear stress.

- **Normal strain:** $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$, $\varepsilon_z = \frac{\partial w}{\partial z}$.
- **Shear strain:** $\gamma_{zx} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$



$$\therefore \varepsilon_x = \frac{O'A' - OA}{OA} = \frac{\partial u}{\partial x}.$$

Similarly,

$$\varepsilon_y = \frac{O'B' - OB}{OB} = \frac{\partial v}{\partial y}.$$

Also,

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

- Strain transformation

$$\begin{Bmatrix} \varepsilon_s \\ \varepsilon_t \\ \frac{\gamma_{st}}{2} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \frac{\gamma_{xy}}{2} \end{Bmatrix}$$

- Principal strain

$$\varepsilon_{I,II} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})}\right)$$

- Max. shear strain $\Rightarrow$ $\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$

$$\theta_3 = \frac{1}{2}\tan^{-1}\left(-\frac{(\varepsilon_{xx} - \varepsilon_{yy})}{\gamma_{xy}}\right)$$

Stress Strain Relationship:

Assumption: 1. Material is isotropic and homogeneous
2. The structure is deformed within elastic limit

Hooke's Law:

When a 1D body (say bar) is loaded axially, it undergoes deformation (ΔL), hence strain and stresses induced. Hooke's law states whenever a body is subjected to external loading within elastic limit, the stresses induced is directly proportional strain, i.e., $\sigma \propto \varepsilon$

$\therefore \boxed{\sigma = E\,\varepsilon}$ ⇒ where, E = Young's Modulus of elasticity.

Poisson's Ratio $(\nu) = -\dfrac{\text{Lateral strain}}{\text{Longitudinal strain}} = -\dfrac{\varepsilon_y}{\varepsilon_x}$

$$\boxed{\nu = -\frac{\varepsilon_y}{\varepsilon_x}}$$

$$\boxed{-1 < \nu < 0.5}$$

Suppose a 3D body is subjected to uniaxial loading in X direction.

$\therefore E = \sigma_x / \varepsilon_x \Rightarrow \varepsilon_x = \frac{\sigma_x}{E} \rightarrow (1)$

Similarly, $\varepsilon_z = \frac{-\nu \sigma_x}{E} \rightarrow (2)$ and $\varepsilon_y = \frac{-\nu \sigma_x}{E} \rightarrow (3)$

Similarly, for uniaxial loading in Y-direction:

$\varepsilon_y = \sigma_y / E$, $\varepsilon_x = -\nu \sigma_y / E$, $\varepsilon_z = \frac{-\nu \sigma_y}{E} \rightarrow (4)$

in Z-direction:

$\varepsilon_z = \sigma_z / E$, $\varepsilon_x = -\nu \sigma_z / E$, $\varepsilon_y = -\nu \sigma_z / E \rightarrow (5)$

For multidirectional loading, combining all equations:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \rightarrow (A)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \rightarrow (B)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \rightarrow (C)$$

- Similar to hooke's law for normal stress, hooke's law in shear component is,

$$\tau \propto \gamma$$

$$\boxed{\tau = G\gamma}$$ G = Shear Modulus