1.2 THRUST FROM A STATIONARY ROCKET ENGINE / MOTOR:

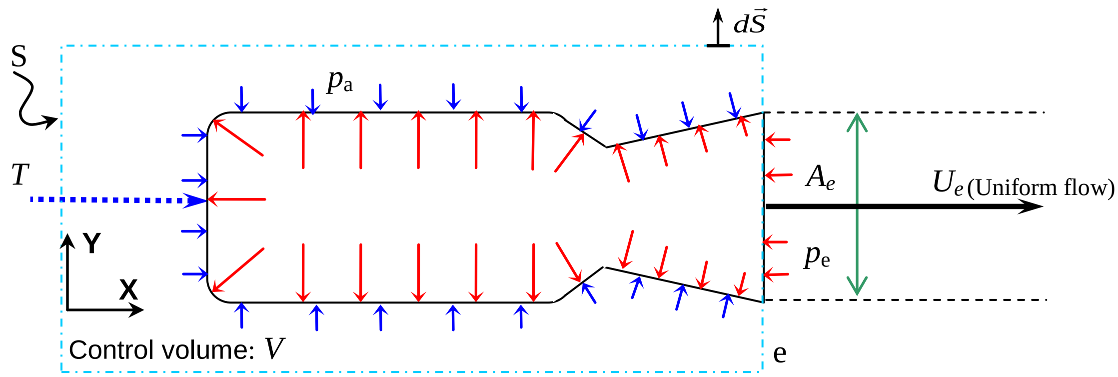

Figure 1 Schematic showing control volume enclosing a stationary rocket motor/engine

Consider a control volume '*V*' (shown in Fig.1) bounded by the control surface 'S'. The control volume is chosen such that a part of control surface 'S' lies on the nozzle exit plane.

Continuity Equation:

Applying the continuity equation to the above control volume 'V'

$$\frac{\partial}{\partial t}\iiint_{CV} \rho dV + \oiint_{CS} \rho \vec{U} \cdot d\vec{S} = 0 \Rightarrow \frac{dM_V}{dt} + \rho_e U_e A_e = 0$$

Rate of change of mass inside control volume *V* — Mass flux through the control surfaces

$$\dot{m}_e = \rho_e U_e A_e = -\dot{M}_V$$

X-Momentum Equation

Assuming the velocity is dominant in X direction and applying X-momentum equation over the control volume 'V',

$$\frac{\partial}{\partial t}\iiint_{CV} u\rho dV + \oiint_{CS} u\rho \vec{U} \cdot d\vec{S} = \sum F_x$$

Rate of change of momentum inside control volume *V* — Momentum flux through the control surfaces — External forces on the system

$$\sum F_X = T + \oiint_S -P d\vec{S} = T + A_e p_a - A_e p_e$$

Thrust (T):

$$T = \dot{m}_e U_e + A_e\left(p_e - p_a\right) = \dot{m}_e U_{eq}$$

where $U_{eq} = U_e + \left(\frac{p_e - p_a}{\dot{m}}\right) A_e$

Total impulse (I):

The total impulse *I* is the thrust force T (which can vary with time) integrated over the burning time t.

$$I = \int T\, dt = \int \dot{m}_e U_{eq} dt = m_p U_{eq} \quad units: \left(kg \frac{m}{s}\right)$$

where m_p = total mass of propellant expelled.

Specific impulse (I_{sp}):

The specific impulse I_{SP} is the total impulse per unit weight of propellant.

$$I_{sp} = \frac{I}{m_p g_e} = \frac{U_{eq}}{g_e} \quad units: \left(\frac{m/s}{m/s^2} \equiv s\right)$$

Here g_e is the acceleration due to gravity at the earth's surface. Note that the choice of g_e is arbitrary. The advantage is that in all common systems (fps, cgs, SI etc.) the unit of specific impulse (I_{sp}) is the same '*seconds*'.

<u>Maximum Thrust:</u>

At this point we ask ourselves; "At what p_e do we get maximum thrust?" It can be seen from Fig.3 that the inner walls encounter higher pressure (due to the combustion in the chamber) compared to outer walls. The component of force acting in the forward direction (w.r.t the rocket motor) contributes to thrust, whereas the component in rearward direction reduces thrust.

Since the chamber pressure ($p_{comb.chamber}$) or the pressure acting on the inner walls decreases towards the exit and the nozzle section at the position where exit pressure $p_e = p_a$ (optimal expansion) is crucial (as this will be seen later). A nozzle longer than this point results in net force in negative thrust direction due to over-expansion (as the pressure acting on the inner walls is less than p_a). The nozzle shorter than the optimal expansion position would result in a net force in positive thrust direction due to under-expansion (as the pressure acting on the inner walls is higher than p_a). But it should be noted that the exiting momentum is not fully regained from p_e in case of shorter nozzle. Finally, it can be seen that a nozzle designed for optimal expansion would result in higher thrust.

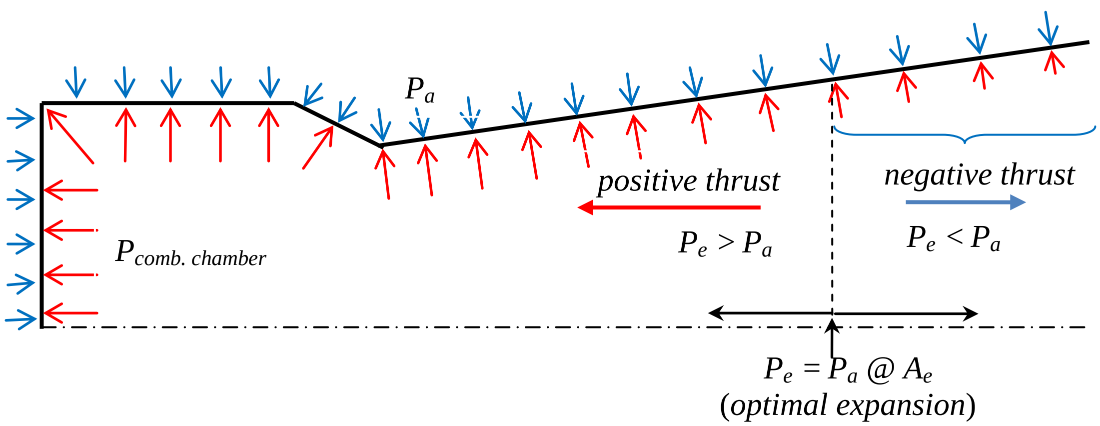

Figure 2. Schematic showing optimal expansion in a CD nozzle.

The thrust equation of the rocket nozzle (as given previously) is

$$T = \dot{m}_e U_e + A_e(p_e - p_a)$$

Differentiating the above equation (noting that U_e & p_e to be the variables), we get

$$dT = \dot{m}_e dU_e + dA_e(p_e - p_a) + A_e dp_e = 0$$

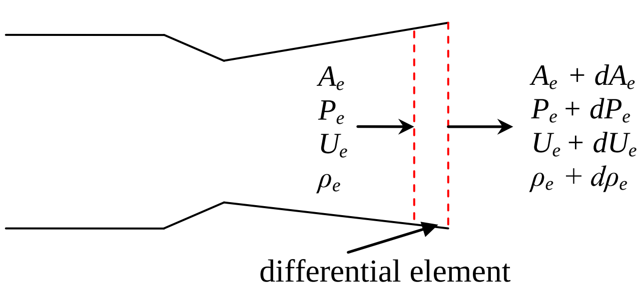

Figure 3 A differential element at the nozzle exit

Momentum equation for a differential element (Fig.2) at exit gives

$$(\rho_e + d\rho_e)(U_e + dU_e)^2(A_e + dA_e) - \rho_e U_e^2 A_e = \sum F_x = A_e P_e - (A_e + dA_e)(P_e + dP_e) + P_e dA_e$$

$$(\rho_e+d\rho_e)(U_e+dU_e)\ (A_e+dA_e)U_e\text{-}\rho_e U_e^2 A_e+(\rho_e+d\rho_e)(U_e+dU_e)\ (A_e+dA_e)dU_e=\sum F_x$$

$$U_e\Big((\rho_e+d\rho_e)(U_e+dU_e)\ (A_e+dA_e)\text{-}\rho_e U_e A_e\Big)+(\rho_e+d\rho_e)(U_e+dU_e)\ (A_e+dA_e)dU_e=\sum F_x$$

Now from continuity,

$$(\rho_e+d\rho_e)(U_e+dU_e)\ (A_e+dA_e)\text{-}\rho_e U_e A_e=0$$

$$\rho_e U_e A_e dU_e + A_e dp_e = \dot{m}_e dU_e + A_e dp_e = 0$$

$dT = dA_e(p_e - p_a) = 0$, or $p_e = p_a$

$$d^2T = d^2A_e(p_e - p_a) + dA_e dp_e$$

$$d^2T\Big|_{at\ p_e=p_a} = dA_e dp_e$$

Now $dA_e dp_e < 0$ as $dA_e > 0\ and\ dp_e < 0$

Therefore, $p_e = p_a$ refers to optimum expansion. Under-expansion $p_e > p_a$ implies additional force is remaining unused and over-expansion $p_e < p_a$ can be noted as the beginning of negative contribution to thrust generation.

Thrust is max. for given chamber condition at optimum expansion i.e. $p_e = p_a$

$$T_{\max} = m_p U_e$$

In normal situation nozzle is not always optimally expanded because p_a is changing with altitude.

<u>Exhaust gas velocity (U_e) in terms of combustion chamber properties</u>

Ideal analysis: one dimensional, steady state, isentropic flow. The gas is assumed to be a perfect gas with constant properties

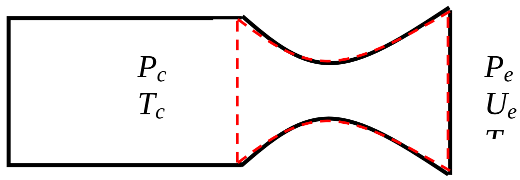

Figure 4 A schematic showing control volume to relate exhaust velocity (*U_e*) to thermodynamic properties in the combustion chamber.

Recall equation for conservation of energy

$$\frac{\partial}{\partial t}\iiint_{CV}\left(e_0+\frac{U^2}{2}\right)\rho dV+\iint_{CS}\left(h+\frac{U^2}{2}\right)\rho\vec{U}\cdot d\vec{S}\quad=\delta\dot{Q}-\delta\dot{W}'$$

$$h_c+\frac{V_{c1}^2}{2}=h_e+\frac{U_e^2}{2}$$

$$2C_p(T_c-T_e)=U_e^2-V_c^2 \qquad (1)$$

$C_p-C_v=R,\quad \frac{C_p}{C_v}=\gamma$ and $R=\frac{R_u}{M_W}$

Eliminating C_v we get $C_p=R\frac{\gamma}{\gamma-1}=\frac{R_u}{M_W}\times\frac{\gamma}{\gamma-1}$

For *isentropic flow* the pressure temperature relationship follows

$$\frac{T_e}{T_c}=\left(\frac{p_e}{p_c}\right)^{\frac{r-1}{r}}\text{, or } T_c-T_e=T_c\left[1-\left(\frac{p_e}{p_c}\right)^{\frac{r-1}{r}}\right] \qquad (2)$$

Substituting for (T_c-T_e) in (1)

$$U_e=\left(\frac{T_c}{M_W}\times\frac{2R_u\gamma}{\gamma-1}\left[1-\left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]+V_c^2\right)^{1/2} \qquad (3)$$

Now

V_c = gas flow velocity in combustion chamber is very small and hence neglected

p_e = outside atmospheric pressure $<< p_c \Rightarrow \frac{p_e}{p_c}$ is very small.

$$\therefore U_e = \text{const} \times \sqrt{\frac{T_c}{M_W}} \tag{4}$$

Note: Exhaust velocity, U_e and thus the specific impulse, I_{sp} is directly proportional to combustion temperature, T_c and inversely proportional to the molecular weight, M_W of combustion gases these are important for selecting propellant.

Specific Impulse (I_{sp}) of representative space propulsion system

Propulsion system type	Working fluid	Specific impulse
Chemical (liquid)		
a) Monopropellant	Hydrogen peroxide	110-140
b) Bipropellant	Hydrazine	220-245
	O_2-H_2	440-480
	O_2-hydrocarbon	340-380
	N_2O_2-Monomethyl hydrazine	300-340
Chemical (solid)	Fuel and oxidizer	260-300
Nuclear	H_2	600-1000
Solar heating	H_2	400-800
Electric (Arcjet)	H_2	400-2000
Electric (Ion)	Cesium	5000-25000
Cold gas	N_2	50-60

Further reading

1. Mechanics and Thermodynamics of Propulsion by Philip.G.Hill & Carl.Peterson. Chapter #10 (Section 10.2)

2. Rocket Propulsion Elements by G. P. Sutton and Oscar Biblarz, Chapter #3, Nozzle Theory and Thermodynamics Relations.